Appendix: Some useful facts

A1. Some handy inequalities. The following inequalities involving exponentials and logarithms are very handy.

(i) For all real numbers x, we have

$$1 + x \le e^x,$$

or, taking logarithms, for $x > -1$, we have

$$\log(1 + x) \le x.$$

(ii) For all real numbers $x \ge 0$, we have

$$e^{-x} \le 1 - x + x^2/2,$$

or, taking logarithms,

$$-x \le \log(1 - x + x^2/2).$$

(iii) For all real numbers x with $0 \le x \le 1/2$, we have

$$1 - x \ge e^{-x-x^2} \ge e^{-2x},$$

or, taking logarithms,

$$\log(1 - x) \ge -x - x^2 \ge -2x.$$

(i) and (ii) follow easily from Taylor's formula with remainder, applied to the function e^x, while (iii) may be proved by expanding $\log(1 - x)$ as a Taylor series, and making a simple calculation.

A2. Binomial coefficients. For integers n and k, with $0 \le k \le n$, one defines the **binomial coefficient**

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

We have the identities

$$\binom{n}{n} = \binom{n}{0} = 1,$$

and for $0 < k < n$, we have **Pascal's identity**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

which may be verified by direct calculation. From these identities, it follows that $\binom{n}{k}$ is an integer, and indeed, is equal to the number of subsets of $\{1, \ldots, n\}$ of cardinality k. The usual **binomial theorem** also follows as an immediate consequence: for all numbers a, b, and for all positive integers n, we have the **binomial expansion**

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

It is also easily verified, directly from the definition, that

$$\binom{n}{k} < \binom{n}{k+1} \quad \text{for } 0 \le k < (n-1)/2,$$

$$\binom{n}{k} > \binom{n}{k+1} \quad \text{for } (n-1)/2 < k < n, \text{ and}$$

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{for } 0 \le k \le n.$$

In other words, if we fix n, and view $\binom{n}{k}$ as a function of k, then this function is increasing on the interval $[0, n/2]$, decreasing on the interval $[n/2, n]$, and its graph is symmetric with respect to the line $k = n/2$.

A3. Countably infinite sets. Let $\mathbb{Z}^+ := \{1, 2, 3, \ldots\}$, the set of positive integers. A set S is called **countably infinite** if there is a bijection $f : \mathbb{Z}^+ \to S$; in this case, we can enumerate the elements of S as $x_1, x_2, x_3, \ldots$, where $x_i := f(i)$.

A set S is called **countable** if it is either finite or countably infinite.

For a set S, the following conditions are equivalent:

- S is countable;
- there is a surjective function $g : \mathbb{Z}^+ \to S$;
- there is an injective function $h : S \to \mathbb{Z}^+$.

The following facts can be easily established:

(i) if $S_1, \ldots, S_n$ are countable sets, then so are $S_1 \cup \cdots \cup S_n$ and $S_1 \times \cdots \times S_n$;

(ii) if $S_1, S_2, S_3, \ldots$ are countable sets, then so is $\bigcup_{i=1}^{\infty} S_i$;

(iii) if S is a countable set, then so is the set $\bigcup_{i=0}^{\infty} S^{\times i}$ of all finite sequences of elements in S.

Some examples of countably infinite sets: $\mathbb{Z}$, $\mathbb{Q}$, the set of all finite bit strings. Some examples of uncountable sets: $\mathbb{R}$, the set of all infinite bit strings.

A4. Integrating piece-wise continuous functions. In discussing the Riemann integral $\int_a^b f(t)\,dt$, many introductory calculus texts only discuss in any detail the case where the integrand f is continuous on the closed interval $[a, b]$, in which case the integral is always well defined. However, the Riemann integral is well defined for much broader classes of functions. For our purposes in this text, it is convenient and sufficient to work with integrands that are **piece-wise continuous** on $[a, b]$, which means that there exist real numbers $x_0, x_1, \ldots, x_k$ and functions $f_1, \ldots, f_k$, such that $a = x_0 \leq x_1 \leq \cdots \leq x_k = b$, and for each $i = 1, \ldots, k$, the function f_i is continuous on the *closed* interval $[x_{i-1}, x_i]$, and agrees with f on the *open* interval (x_{i-1}, x_i). In this case, f is integrable on $[a, b]$, and indeed

$$\int_a^b f(t)\,dt = \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} f_i(t)\,dt.$$

It is not hard to prove this equality, using the basic definition of the Riemann integral; however, for our purposes, we can also just take the value of the expression on the right-hand side as the definition of the integral on the left-hand side.

If f is piece-wise continuous on $[a, b]$, then it is also bounded on $[a, b]$, meaning that there exists a positive number M such that $|f(t)| \leq M$ for all $t \in [a, b]$, from which it follows that $|\int_a^b f(t)\,dt| \leq M(b - a)$.

We also say that f is piece-wise continuous on $[a, \infty)$ if for all $b \geq a$, f is piece-wise continuous on $[a, b]$. In this case, we may define the improper integral $\int_a^{\infty} f(t)\,dt$ as the limit, as $b \to \infty$, of $\int_a^b f(t)\,dt$, provided the limit exists.

A5. Estimating sums by integrals. Using elementary calculus, it is easy to estimate a sum over a monotone sequence in terms of a definite integral, by interpreting the integral as the area under a curve. Let f be a real-valued function that is (at least piece-wise) continuous and monotone on the closed interval $[a, b]$, where a and b are integers. Then we have

$$\min(f(a), f(b)) \le \sum_{i=a}^{b} f(i) - \int_a^b f(t)\,dt \le \max(f(a), f(b)).$$

A6. Infinite series. Consider an infinite series $\sum_{i=1}^{\infty} x_i$. It is a basic fact from calculus that if the x_i's are non-negative and $\sum_{i=1}^{\infty} x_i$ converges to a value y, then any infinite series whose terms are a rearrangement of the x_i's converges to the same value y.

If we drop the requirement that the x_i's are non-negative, but insist that the series $\sum_{i=1}^{\infty} |x_i|$ converges, then the series $\sum_{i=1}^{\infty} x_i$ is called **absolutely convergent**. In this case, then not only does the series $\sum_{i=1}^{\infty} x_i$ converge to some value y, but any infinite series whose terms are a rearrangement of the x_i's also converges to the same value y.

A7. Double infinite series. The topic of **double infinite series** may not be discussed in a typical introductory calculus course; we summarize here the basic facts that we need.

Suppose that $\{x_{ij}\}_{i,j=1}^{\infty}$ is a family non-negative real numbers such that for each i, the series $\sum_j x_{ij}$ converges to a value r_i, and for each j the series $\sum_i x_{ij}$ converges to a value c_j. Then we can form the double infinite series $\sum_i \sum_j x_{ij} = \sum_i r_i$ and the double infinite series $\sum_j \sum_i x_{ij} = \sum_j c_j$. If $(i_1, j_1), (i_2, j_2), \ldots$ is an enumeration of all pairs of indices (i, j), we can also form the single infinite series $\sum_k x_{i_k j_k}$. We then have $\sum_i \sum_j x_{ij} = \sum_j \sum_i x_{ij} = \sum_k x_{i_k j_k}$, where the three series either all converge to the same value, or all diverge. Thus, we can reverse the order of summation in a double infinite series of non-negative terms. If we drop the non-negativity requirement, the same result holds provided $\sum_k |x_{i_k j_k}| < \infty$.

Now suppose $\sum_i a_i$ is an infinite series of non-negative terms that converges to A, and that $\sum_j b_j$ is an infinite series of non-negative terms that converges to B. If $(i_1, j_1), (i_2, j_2), \ldots$ is an enumeration of all pairs of indices (i, j), then $\sum_k a_{i_k} b_{j_k}$ converges to AB. Thus, we can multiply term-wise infinite series with non-negative terms. If we drop the non-negativity requirement, the same result holds provided $\sum_i a_i$ and $\sum_j b_j$ converge absolutely.

A8. Convex functions. Let I be an interval of the real line (either open, closed, or half open, and either bounded or unbounded), and let f be a real-valued function defined on I. The function f is called **convex on** I if for all $x_0, x_2 \in I$, and for all $t \in [0, 1]$, we have

$$f(tx_0 + (1-t)x_2) \le tf(x_0) + (1-t)f(x_2).$$

Geometrically, convexity means that for every three points $P_i = (x_i, f(x_i))$, $i = 0, 1, 2$, where each $x_i \in I$ and $x_0 < x_1 < x_2$, the point P_1 lies on or below the line through P_0 and P_2.

We state here the basic analytical facts concerning convex functions:

(i) if f is convex on I, then f is continuous on the interior of I (but not necessarily at the endpoints of I, if any);

(ii) if f is continuous on I and differentiable on the interior of I, then f is convex on I if and only if its derivative is non-decreasing on the interior of I.